Pergamon

Int. J. Heat Mass Transfer. Vol. 41, No. 13, pp. 1983–1991, 1998

Printed in Great Britain
0017–9310/98 \$19.00+0.00

PII: S0017–9310(97)00282–2

Natural convective cooling of a horizontal heat conducting plate facing up in an otherwise adiabatic cavity

E. LUNA and C. TREVIÑO†

Facultad de Ciencias, UNAM, 04510 México D.F., Mexico

(*Received* 21 *March* 1997 *and in final form* 17 *September* 1997)

Abstract—The transient conjugate cooling of a thin horizontal plate enclosed in an adiabatic cavity is studied using asymptotic and numerical techniques. The upper surface of the plate is in contact with a fluid initially at rest. The most important parameter to be obtained is the nondimensional cooling time, which depends on the aspect ratio of the plate ε, the nondimensional longitudinal heat conductance of the plate α and the Prandtl number of the fluid Pr, for large values of the Rayleigh number of the laminar natural boundary layer flow induced by the density changes. Both the thermally thin ($\alpha \gg \varepsilon^2$) and thick wall regimes ($\alpha \sim \varepsilon^2$) are considered in this paper. A minimum nondimensional cooling time is achieved for a thin plate ($\varepsilon \to 0$) for values of the nondimensional longitudinal thermal conductance of the plate α, such as $\varepsilon^2 \ll \alpha \ll 1$. That is, the process is faster if the longitudinal heat conduction through the wall is negligible.

1. INTRODUCTION

The study of coupled interaction of conduction and convection heat transfer is extremely important because it appears in many practical and industrial devices. Many works have appeared in the literature studying the free or natural convective heat transfer from horizontal solid surfaces facing up or down, with prescribed surface temperature or heat flux [1–10], among others, since the now classical work of Stewartson [1]. An excellent review can be found in [9]. However, the combined conduction-convection heat transfer process for horizontal plates has received very little attention. The influence of solid heat conduction in horizontal plates on the natural convective heat transfer process has been analyzed only in few works. Luna *et al.* [11] and Luna and Treviño [12] studied the steady-state and the transient process of a horizontal wall separating two fluids at different temperatures, respectively. They found that the longitudinal heat conduction does not have any influence in the steady-process, because of the absence of any temperature gradients in the thermally thin wall regime. However, for the transient process, the longitudinal heat conduction through the plate has an important effect on the evolution time.

The objective of this paper is to evaluate the cooling process of a horizontal heat conducting plate, which has a fluid over its upper face. The other surfaces are assumed to be adiabatic. We will study here the case where the plate is hotter than the fluid. This process can be found in many practical devices, specially in electronic equipment like chips over horizontal surfaces. For very large values of the Rayleigh number, Ra, boundary layers develop from the edges of the plate towards the center, forming here a buoyant plume rising above the upper surface. The thickness of the plume is of the order of magnitude of the thickness of the boundary layer at the center. Here, the boundary layer approximation breaks down. However, for large values of the Rayleigh number, the thickness of this region related to the length of the plate is very small, of order $Ra^{-1/5}$. The central region is for this problem not very important, because the higher heat transfer rates occur on the edges of the plate. For these reasons, it is justified to neglect the influence of the plume in the present work.

2. FORMULATION

Consider a horizontal heat conducting strip of width $2L$, thickness h and initial uniform temperature T_{w0}. The upper face of the strip contacts a fluid with temperature $T_\infty < T_{w0}$. The lower and lateral walls are supposed to be adiabatic. The physical model under study is shown in Fig. 1. Due to the longitudinal heat conduction in the strip, temperature differences appear in the fluid, inducing natural convection flows due to the corresponding density changes. An order of magnitude analysis shows that these motions occur in boundary layers with thickness of order $L/Ra^{1/5}$, for large values of the Rayleigh number, $Ra = g\beta\Delta T\, Pr\, L^3/\nu^2$. Here, g is the acceleration of gravity, β and ν are thermal expansion coefficients and kinematic viscosities of the fluid. Pr denotes the Prandtl number, $Pr = \rho\nu c/\lambda$, where ρ is the density, c is the specific heat

† Author to whom correspondence should be addressed.

NOMENCLATURE

c	specific heat of the fluid
c_w	specific heat of the plate
f	nondimensional stream function for the fluid defined in equation (6)
g	acceleration of gravity
h	thickness of the strip
L	half-width of the strip
Pr	Prandtl number of the fluid, $Pr = \mu c/\lambda$
Ra_∞	Rayleigh number, $Ra = g\beta\Delta T_\infty Pr\, L^3/\nu^2$
s	nondimensional slow time variable, $s = \tau$
T_{w0}	initial temperature of the plate
T_∞	fluid temperature far away from the plate
t_e	characteristic cooling time for the plate
t_d	diffusion time in the transversal direction of the plate
t_{dL}	diffusion time in the longitudinal direction of the plate
t_r	characteristic residence time, $t_r = L/u_c$
u_c	characteristic induced velocity introduced in equation (1)
x, y	horizontal and vertical Cartesian coordinates
z	nondimensional transversal coordinate for the plate, $z = y/h$.

Greek symbols

α	heat conduction parameter defined in equation (3)
β	thermal expansion coefficient of the fluid
δ	thickness of the boundary layer flow
ΔT_∞	initial temperature difference, $\Delta T_\infty = T_{w0} - T_\infty$
ε	aspect ratio of the plate, $\varepsilon = h/L$
η	similar variable defined in equation (5)
θ	nondimensional temperature of the fluid defined in equation (6)
θ_w	nondimensional temperature of the plate defined in equation (6)
λ	thermal conductivity of the fluid
λ_w	thermal conductivity of the plate
μ	dynamic viscosity of the fluid
ν	kinematic viscosity of the fluid
ξ	nondimensional time, $\xi = \alpha\tau/\varepsilon^2$
ρ	density of the fluid
ρ_w	density of the plate
σ	nondimensional short time variable, $\sigma = \alpha\tau$
τ	nondimensional time defined in equation (5)
ϕ	nondimensional pressure defined in equation (6)
χ	nondimensional longitudinal coordinate, $\chi = x/L$
ψ	stream function for the fluid.

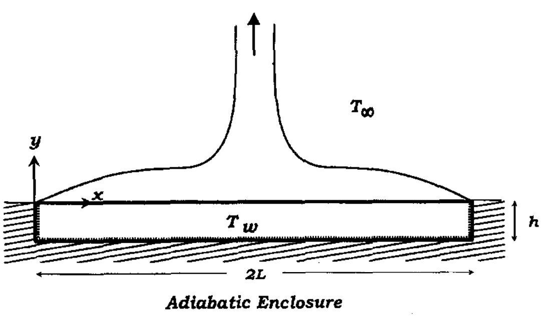

Fig. 1. Physical problem sketch showing the plate in a partial adiabatic enclosure.

and λ is the thermal conductivity of fluid, respectively. ΔT is the actual temperature difference across the fluid layer. After defining the Rayleigh number with the initial temperature difference, $Ra_\infty = g\beta(T_{w0} - T_\infty)Pr\, L^3/\nu^2$, the order of magnitude of the boundary layer thickness and the induced velocity are given by

$$\delta \sim \frac{L}{Ra_\infty^{1/5}}\left(\frac{\Delta T_\infty}{\Delta T}\right)^{1/5} \quad \text{and} \quad u_c \sim \frac{Ra_\infty^{2/5}\nu}{Pr\, L}\left(\frac{\Delta T}{\Delta T_\infty}\right)^{2/5}, \tag{1}$$

where $\Delta T_\infty = T_{w0} - T_\infty$, corresponds to the initial temperature difference. The order of magnitude of the heat flux across the fluid is then

$$q \sim \frac{\lambda(\Delta T)^{6/5} Ra_\infty^{1/5}}{L(\Delta T_\infty)^{1/5}} \sim \frac{\lambda_w \Delta T_w}{h} \sim \frac{\rho_w c_w h \Delta T_\infty}{t_e}. \tag{2}$$

In these relationships, ρ_w, c_w and λ_w represent the density, specific heat and thermal conductivity of the wall material. ΔT_w is the characteristic transversal temperature drop at the wall and t_e is the characteristic evolution time of the transient process. The last term in relation (2) comes from the thermal energy accumulation. From relationships (2), we obtain the order of magnitude of the characteristic evolution time as

$$t_e \sim \frac{\alpha}{\varepsilon^2} t_d \quad \text{with} \quad \alpha = \frac{\lambda_w}{\lambda}\frac{h}{L}\frac{1}{Ra_\infty^{1/5}},$$

$$\varepsilon = \frac{h}{L} \quad \text{and} \quad t_d \sim \frac{h^2 \rho_w c_w}{\lambda_w}. \tag{3}$$

Here, t_d corresponds to the diffusion time in the transversal direction of the strip, ε is to the aspect ratio of the strip and is to be assumed very small compared with unity. Parameter α is the nondimensional longitudinal heat conductance of the trip and corresponds to the ratio of heat conducted longitudinally by the plate to the heat convected to the fluid. This parameter can have values much larger or much smaller than unity, depending on the strip material. For values such as $\alpha/\varepsilon^2 \gg 1$, the evolution time is much larger than the diffusion time in the transversal direction of the plate. Therefore, no large temperature differences in the transversal direction of the strip are allowed. In this regime, the temperature variations in the transversal direction of the strip can be neglected, being very small, of order ε^2/α, compared with the overall temperature difference. That is $\Delta T_w \ll \Delta T$. This regime is called the thermally thin wall regime. The first relation in (3) can also be written in the form $t_e \sim \alpha t_{dL}$, where $t_{dL} \sim L^2\rho_w/c_w\lambda_w$ corresponds to the diffusion time in the longitudinal direction. For small values of α compared with unity, the evolution time is shorter than the longitudinal diffusion time, indicating the existence of large temperature variations in the longitudinal direction. Then α is a measure of the importance of the longitudinal heat conduction in the cooling process. For values of $\alpha/\varepsilon^2 \sim 1$, the evolution time is of order of the transversal diffusion time and then the temperature variations in both directions of the strip now are very important. This regime is called the thermally thick wall regime. In this regime because $\varepsilon \ll 1$, the longitudinal heat conduction through the strip is very small and can be neglected. Due to the singular character of the limit $\alpha \to 0$, the longitudinal heat conduction term is to be retained only in thin layers close to the vertical edges of the strip, in order to achieve the adiabatic boundary condition.

The characteristic residence time in the boundary layer flow t_r, of order $t_r \sim L/u_c$, can be obtained using (1). The quasi-steady approximation for the fluid is fully justified, for large values of the ratio t_e/t_r. This ratio is then given by

$$\frac{t_e}{t_r} \sim \frac{\rho_w c_w}{\rho c} \varepsilon \, Ra_\infty^{1/5} \left[\frac{\Delta T}{\Delta T_\infty}\right]^{2/5} \gg 1. \tag{4}$$

For large values of the Rayleigh number, relation (4) holds. This is the limit studied in this work.

Using the guidance of the order of magnitude estimates, we introduce the following nondimensional independent variables

$$\chi = \frac{x}{L}, \quad \eta = Ra^{1/5}\frac{y}{5^{1/5}L\chi^{2/5}}, \quad z = \frac{y}{h}, \quad \tau = \frac{\varepsilon^2}{\alpha}\frac{t}{t_d}, \tag{5}$$

together with the nondimensional dependent variables

$$f = \frac{\psi \, Pr}{5^{4/5}\,\nu \, Ra^{1/5}\chi^{3/5}}, \quad \phi = \frac{(p-p_0)L\,Pr}{5^{1/5}\rho\nu^2 \, Ra^{4/5}\chi^{2/5}},$$

$$\theta = \frac{T-T_\infty}{T_{w0}-T_\infty}, \quad \theta_w = \frac{T_w - T_\infty}{T_{w0}-T_\infty}. \tag{6}$$

Here x is the horizontal distance from one of the edges of the strip, y is the vertical distance measured from the middle plane of the strip pointing upward, ψ is the stream function defined in the usual way and p is the pressure. The nondimensional balance equations, using the Boussinesq and boundary layer approximations, then take the form

$$\frac{\partial^2\theta}{\partial\eta^2} + 3f\frac{\partial\theta}{\partial\eta} = 5\chi\left(\frac{\partial f}{\partial\eta}\frac{\partial\theta}{\partial\chi} - \frac{\partial f}{\partial\chi}\frac{\partial\theta}{\partial\eta}\right) \tag{7}$$

$$\frac{\partial\phi}{\partial\eta} = \theta \tag{8}$$

$$\frac{\partial^3 f}{\partial\eta^3} + \frac{2}{5}\eta\frac{\partial\phi}{\partial\eta} - \frac{2}{5}\phi - \chi\frac{\partial\phi}{\partial\chi}$$
$$= \frac{1}{Pr}\left[\left(\frac{\partial f}{\partial\eta}\right)^2 - 3f\frac{\partial^2 f}{\partial\eta^2} + 5\chi\left(\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\chi\,\partial\eta} - \frac{\partial f}{\partial\chi}\frac{\partial^2 f}{\partial\eta^2}\right)\right] \tag{9}$$

for the fluids and

$$\alpha\frac{\partial^2\theta_w}{\partial\chi^2} + \frac{\alpha}{\varepsilon^2}\frac{\partial^2\theta_w}{\partial z^2} = \frac{\partial\theta_w}{\partial\tau} \tag{10}$$

for the strip. The initial condition is $\theta_w(\chi, z, \tau = 0) = 1$, while the boundary conditions are given by

$$f = \frac{\partial f}{\partial\eta} = \theta - \theta_w = \frac{\partial\theta_w}{\partial z} - \frac{\varepsilon^2}{\alpha\chi^{2/5}}\frac{\partial\theta}{\partial\eta} = 0$$
$$\text{at } \eta = z - \frac{1}{2} = 0 \tag{11}$$

$$\frac{\partial\theta_w}{\partial z} = 0 \quad \text{at } z = -\frac{1}{2} \tag{12}$$

$$\frac{\partial\theta_w}{\partial\chi} = 0 \quad \text{for } \chi = 0 \quad \text{and} \quad \chi = 1 \tag{13}$$

$$\frac{\partial f}{\partial\eta} = \theta = \phi = 0 \quad \text{for } \eta \to \infty. \tag{14}$$

In general this system of elliptic equations can be numerically integrated. In the following section we explore asymptotic solutions in both, the thermally thin and thick wall regimes.

3. THERMALLY THIN WALL REGIME

As mentioned before, for very large values of α/ε^2 compared with unity, the temperature variations in the transversal direction in the wall can be neglected. In this regime, the nondimensional energy equation for the wall (10) can be integrated along the transversal coordinate, resulting after applying the boundary conditions (11) and (12)

$$\alpha \frac{\partial^2 \theta_w}{\partial \chi^2} - \frac{\partial \theta_w}{\partial \tau} = -\frac{1}{\chi^{2/5}} \frac{\partial \theta}{\partial \eta}\bigg|_{\eta=0}. \quad (15)$$

This equation must be solved with initial condition $\theta_w(\chi, \tau = 0) = 1$ (another different initial condition can be introduced without difficulty), together with the adiabatic conditions for the lateral surface of the wall given by equations (13). In the following subsection we present the asymptotic solution for $\alpha \gg 1$, for this thermally thin wall regime. For values of α order unity, the problem must be solved numerically.

3.1. *Asymptotic limit* $\alpha \gg 1$

For very large values of the parameter α, the nondimensional temperature of the plate changes very little in the longitudinal direction, of order α^{-1}. Two time scales appear in this problem. A slow time scale of order unity in τ controlling the global transient evolution response of the plate and a rapid time scale of order α^{-1} at the beginning of the process. This rapid transient is due to the initial condition of the uniform plate temperature. In the solid–fluid interface, the heat flux to the ambient fluid decreases downstream, because the boundary layer is thicker there, thus generating important longitudinal temperature gradients. Heat is then diffused downstream in the strip in a short time of order α^{-1}, reaching the conditions for a slow further evolution of the plate temperature. For a thermally thin wall, this conjugate heat transfer problem can be studied using multiple scale analysis in the asymptotic limit $\alpha \to \infty$, assuming the following expansions

$$\theta_w = \sum_{j=0}^{\infty} \frac{1}{\alpha^j} \theta_{wj}(\chi, s, \sigma), \quad \Omega = \sum_{j=0}^{\infty} \frac{1}{\alpha^j} \Omega_j(\chi, \eta : s) \quad (16)$$

where s and σ are the slow and fast time scales defined as follows:

$$s = \tau \quad \text{and} \quad \sigma = \alpha\tau, \quad (17)$$

with Ω corresponding to any variable of the fluid, like θ, f or ϕ. Introducing the above relationships (16) and (17) into the non-dimensional governing equation for the plate (15), we obtain the following set of equations

$$\frac{\partial^2 \theta_{wi}}{\partial \chi^2} - \frac{\partial \theta_{wi}}{\partial \sigma} = b_i \left[\frac{\partial \theta_{w(i-1)}}{\partial s} - \frac{1}{\chi^{2/5}} \frac{\partial \theta_{(i-1)}}{\partial \eta}\bigg|_{\eta=0} \right] \quad \text{for all } i, \quad (18)$$

where $b_0 = 0$ and $b_i = 1$ for $i > 0$. The problem is to be solved with the following initial and boundary conditions

$$\theta_{w0}(\chi, 0, 0) = 1, \theta_{wi}(\chi, 0, 0) = 0 \quad \text{for } i > 0,$$
$$\frac{\partial \theta_{wi}}{\partial \chi} = 0 \quad \text{at } \chi = 0, 1 \quad \text{for all } i. \quad (19)$$

Equations (18) can be integrated from $\chi = 0$ to $\chi = 1$ and after applying the adiabatic boundary conditions at both edges, we obtain

$$\frac{\partial}{\partial \sigma} \int_0^1 \theta_{wi} \, d\chi = -b_i \left[\frac{\partial}{\partial s} \int_0^1 \theta_{w(i-1)} \, d\chi - \int_0^1 \frac{d\chi}{\chi^{2/5}} \frac{\partial \theta_{(i-1)}}{\partial \eta}\bigg|_{\eta=0} \right] = 0 \quad \text{for all } i. \quad (20)$$

The right-hand side of equations (20) has to be zero in order to avoid having secular terms for θ_{wi}. The leading order variable θ_{w0} cannot be a function of the short time scale σ and also the longitudinal coordinate χ, because $b_0 = 0$. Thus, θ_{w0} depends only on the large time scale s, $\theta_{w0} = \theta_{w0}(s)$. This function can be found after integrating the first order equation (20), with the corresponding adiabatic conditions at both edges, giving the first order differential equation for θ_{w0} as

$$\frac{5}{3} \frac{d\theta_0}{d\eta}\bigg|_{\eta=0} - \frac{d\theta_{w0}}{ds} = 0. \quad (21)$$

Using the invariance of the boundary layer equations under the group of transformations

$$\theta \Rightarrow B\theta, \quad \eta \Rightarrow B^{-1/5}\eta, \quad \phi \Rightarrow B^{4/5}\phi, \quad f \Rightarrow B^{1/5}f, \quad (22)$$

we can normalize them using $B = \theta_{w0}$. The problem then reduces to the classical heat transfer problem with a surface nondimensional temperature of unity. Therefore [9]

$$\frac{\partial \theta_0}{\partial \eta}\bigg|_{\eta=0} = -G_0 \theta_{w0}^{6/5}, \quad (23)$$

where G_0 is the fluid nondimensional temperature gradient at the wall for the normalized case, $G_0(Pr) \approx 0.394\ Pr^{1/20}$, for Prandtl numbers close to unity [10]. Using equations (21) and (23) with the initial condition $\theta_{w0}(0) = 1$ we obtain the leading term of the asymptotic solution for θ_w as

$$\theta_{w0} = \frac{1}{\left(1 + \frac{G_0}{3} s\right)^5}. \quad (24)$$

Introducing the solution for θ_{w0} into the first order equation (18), this takes the form

$$\frac{\partial^2 \theta_{w1}}{\partial \chi^2} - \frac{\partial \theta_{w1}}{\partial \sigma} = \frac{d\theta_{w0}}{ds} \left[1 - \frac{3}{5\chi^{2/5}} \right] \quad (25)$$

with the initial and boundary conditions given by equation (19). This equation can be more simplified by introducing the function $\varphi(\chi, \sigma)$ in the following form

$$\theta_{w1} = \frac{d\theta_{w0}}{ds}\left[\frac{1}{2}\chi^2 - \frac{5}{8}\chi^{8/5} + \varphi(\chi, s, \sigma)\right].$$

Therefore, equation (25) takes the homogeneous form

$$\frac{\partial^2 \varphi}{\partial \chi^2} - \frac{\partial \varphi}{\partial \sigma} = 0. \quad (26)$$

The initial and boundary conditions are

$$\varphi(\chi, 0, 0) = F(\chi) = -\frac{1}{2}\chi^2 + \frac{5}{8}\chi^{8/5},$$

$$\frac{\partial \varphi}{\partial \chi} = 0, \quad \text{at } \chi = 0 \quad \text{and} \quad \chi = 1. \quad (27)$$

Using the Fourier transform technique, the solution of the equation (26) can be written as

$$\varphi(\chi, s, \sigma) = \sum_{n=0}^{\infty} [A_n(s)\cos n\pi\chi]\exp[-n^2\pi^2\sigma] \quad (28)$$

where the initial conditions for the functions $A_i(s)$ are given by

$$A_0(0) = \int_0^1 F(\chi)\,d\chi = \frac{23}{312},$$

$$A_n(0) = 2\int_0^1 F(\chi)\cos n\pi\chi\,d\chi \quad \text{for } n = 1, 2, \ldots. \quad (29)$$

Therefore, the solution to equation (10) is finally given as

$$\theta_{w1} = \frac{d\theta_{w0}}{ds}\left[\frac{1}{2}\chi^2 - \frac{5}{8}\chi^{8/5} + A_0(s) + \sum_{n=1}^{\infty}[A_n(s)\cos n\pi\chi]\exp[-n^2\pi^2\sigma]\right]. \quad (30)$$

The nondimensional temperature gradient to the first order in the fluid is

$$\left.\frac{\partial\theta_1}{\partial\eta}\right|_{\eta=0} = -\frac{d\theta_{w0}}{ds}\theta_{w0}^{1/5}\left(\frac{1}{2}G_1(2)\chi^2 - \frac{5}{8}G_1(8/5)\chi^{8/5} + A_0(s)G_1(0) + D\right)$$

where D has the rapid vanishing terms containing $\exp[-n^2\pi^2\sigma]$. The functions $G_1(m)$ are plotted in Fig. 2, representing the nondimensional temperature gradients for the linearized version of equations (7)–(9), with the conditions at the wall given by $\theta_{w0} = 1$ and $\theta_{w1} = \chi^m$. Repeating again the procedure, the second order equation in (20) from $\chi = 0$ to $\chi = 1$, we finally obtain

$$-\frac{\partial}{\partial\sigma}\int_0^1 \theta_{w2}\,d\chi - edt = \frac{d\theta_{w0}}{ds}\left[\left(\frac{5G_1(2)}{26} - \frac{25G_1(8/5)}{88} + \frac{5A_0(s)G_1(0)}{3}\right)\theta_{w0}^{1/5}\right.$$

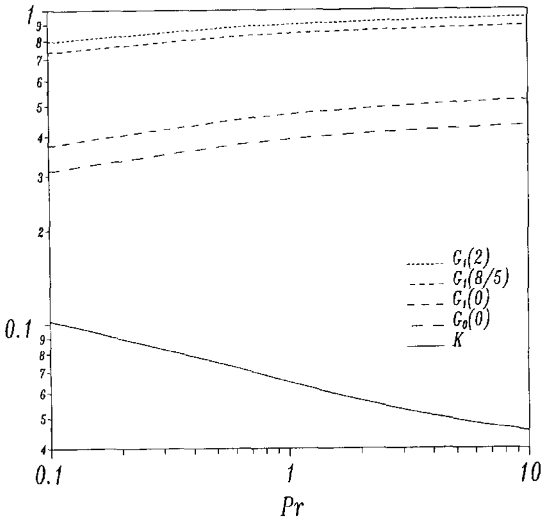

Fig. 2. Values of the nondimensional temperature gradients $G_0(Pr)yG_1(n, Pr)$, for different values of n, as a function of the Prandtl number. The function $K(Pr)$ is also plotted.

$$\left.-G_0\theta_{w0}^{1/5}\left(2A_0(s) - \frac{23}{156}\right) + \frac{dA_0(s)}{ds}\right] = 0 \quad (31)$$

where *edt* means the exponentially decaying terms. In this case we obtain an ordinary differential equation for $A_0(s)$, to be solved with the initial condition $A_0(0) = \int_0^1 F(\chi)\,d\chi$. The solution is given by

$$A_0(s) = A_0(0) + K(Pr)\ln\left[1 + \frac{G_0}{3}s\right]$$

where

$$K(Pr) = -\frac{69}{156} - \frac{15}{26}\frac{G_1(2)}{G_0} + \frac{75}{88}\frac{G_1(8/5)}{G_0}.$$

For a Prandtl number of unity, we obtain $K(1) \simeq 0.06511$. Figure 2 also shows $K(Pr)$. For all possible Prandtl numbers, $K(Pr)$ is always positive.

Up to the first order, the nondimensional temperature at the plate takes the form

$$\theta_w = \left(1 + \frac{G_0}{3}s\right)^{-5} - \frac{1}{\alpha}\frac{5G_0}{3}\left(1 + \frac{G_0}{3}s\right)^{-6} \times \left\{\frac{1}{2}\chi^2 - \frac{5}{8}\chi^{8/5} + \frac{23}{312} + K(Pr)\ln\left[1 + \frac{G_0}{3}s\right] + \sum_{n=1}^{\infty} A_n(s)\exp(-n^2\pi\sigma)\cos n\pi\chi\right\} + O(\alpha^{-2}). \quad (32)$$

Finally, the corresponding value of the overall nondimensional thermal energy of the plate is then

$$\overline{\theta_w} = \int_0^1 \theta_w\,d\chi = \left(1 + \frac{G_0}{3}\tau\right)^{-5}$$

$$-\frac{5G_0K(Pr)}{3\alpha}\left(1+\frac{G_0}{3}\tau\right)^{-6}\ln\left[1+\frac{G_0}{3}\tau\right]+O(\alpha^{-2}). \tag{33}$$

The transient evolution nondimensional time needed to reach any prescribed value of the overall thermal energy, $\overline{\theta}_w$, can be approximated by

$$\frac{\tau}{\tau^*} \simeq 1-\frac{K(Pr)}{\alpha\tau^*}\ln\left[1+\frac{G_0}{3}\tau^*\right], \tag{34}$$

where τ^* corresponds to the value for $\alpha \to \infty$. For $Pr = 1$ and $\overline{\theta}_w = 0.25$, $\tau^* \simeq 2.4328$ and $\tau/\tau^* \simeq 1-0.00742/\alpha$. Due to the fact that $K(Pr)$ is always positive, the transient process is faster for decreasing values of α. This means that reducing the importance of the longitudinal heat conduction in the wall (for example by decreasing the thermal conductivity of the strip material) makes faster the cooling process in this regime. Therefore, it must be a minimum nondimensional cooling time in the merging region of both, the thermally thin and thick wall regimes.

4. THERMALLY THICK WALL REGIME

In this regime, the longitudinal heat conduction is also very small and is to be neglected. The energy balance equation for the plate then reduces to

$$\frac{\partial^2\theta_w}{\partial z^2} = \frac{\partial\theta_w}{\partial\xi}, \tag{35}$$

where $\xi = \alpha\tau/\varepsilon^2$ is the appropriate nondimensional time for the thermally thick wall regime. Equation (35) has to be solved with the initial and boundary conditions:

$$\theta_w(\chi, z, 0) = 0, \quad \frac{\partial\theta_w}{\partial z} = 0 \quad \text{at } z = -1,$$

$$\frac{\partial\theta_w}{\partial z} = \frac{\varepsilon^2}{\alpha\chi^{2/5}}\frac{\partial\theta}{\partial\eta} \quad \text{at } \eta = z-1 = 0. \tag{36}$$

To study the asymptotic limit of $\alpha/\varepsilon^2 \to 0$, we introduce the following regular perturbation expansion for the nondimensional variables, given by:

$$\theta_w(\chi, z, \xi) = \theta_{w0}(\chi, z, \xi)+\frac{\alpha}{\varepsilon^2}\theta_{w1}(\chi, z, \xi)+O\left[\left(\frac{\alpha}{\varepsilon^2}\right)^2\right] \tag{37}$$

$$\Omega(\chi, \eta : \xi) = \Omega_0(\chi, \eta : \xi)+\frac{\alpha}{\varepsilon^2}\Omega_1(\chi, \eta : \xi)+O\left[\left(\frac{\alpha}{\varepsilon^2}\right)^2\right], \tag{38}$$

where Ω corresponds to any variable of fluid, like θ, f or ϕ. In this form, we obtain up to terms of order α/ε^2 the following set of equations:

$$\frac{\partial^2\theta_{wi}}{\partial z^2} = \frac{\partial\theta_{wi}}{\partial\xi} \quad \text{for all } i \tag{39}$$

with the following initial and boundary conditions

$$\theta_{w0}(\chi, z, 0) = 1, \quad \left.\frac{\partial\theta_0}{\partial\eta}\right|_{\eta=0} = 0$$

$$\left.\frac{1}{\chi^{2/5}}\frac{\partial\theta_j}{\partial\eta}\right|_{\eta=0} = \left.\frac{\partial\theta_{w(j-1)}}{\partial z}\right|_{z=0},$$

$$\left.\frac{\partial\theta_{w(j-1)}}{\partial z}\right|_{z=-1} = 0, \quad \theta_{wi}(\chi, z, 0) = 0 \quad \text{for } i > 0. \tag{40}$$

The leading order solution for θ_0 is the trivial solution $\theta_0 = 0$. The solution to equation (39) for θ_{w0} can then be written as

$$\theta_{w0}(\chi, z, \xi) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n-1/2}$$

$$\times\cos((n-1/2)\pi(z+1))\exp(-(n-1/2)^2\pi^2\xi). \tag{41}$$

Introducing the solution (41) into the boundary conditions (40), the first order equations transform to

$$\frac{\partial^2\theta_{w1}}{\partial z^2} = \frac{\partial\theta_{w1}}{\partial\xi}, \tag{42}$$

with the initial and boundary conditions

$$\theta_{w1}(\chi, z, 0) = 0,$$

$$\left.\frac{1}{\chi^{2/5}}\frac{\partial\theta_1}{\partial\eta}\right|_{\eta=0} = -H(\xi), \quad \left.\frac{\partial\theta_{w1}}{\partial z}\right|_{z=-1} = 0. \tag{43}$$

The solution to the first order in the fluid is then given by

$$\theta_{w1}(\chi, 0, \xi) = \chi^{1/3}\left(\frac{H(\xi)}{G(1/3)}\right)^{5/6}. \tag{44}$$

Here, the function $H(\xi)$ is given by

$$H(\xi) = 2\sum_{n=1}^{\infty}\exp(-(n-1/2)^2\pi^2\xi).$$

The fluid balance equations are to be solved using the conditions given in equation (43). This in fact represents the classical problem with a known heat flux. The solution can be obtained by assuming

$$\theta_1 = B\chi^{1/3}\varphi, \quad f = B^{1/5}\chi^{1/15}\bar{f},$$

$$\phi = B^{4/5}\chi^{4/15}\bar{\phi} \quad \text{and} \quad \eta = B^{-1/5}\chi^{-1/15}\bar{\eta} \tag{45}$$

with

$$B = \left(\frac{H(\xi)}{G_{1/3}(Pr)}\right)^{5/6}$$

where $G_{1/3}(Pr)$ is the nondimensional temperature gradient at $\bar{\eta} = 0$, for the reduced normalized problem with $\varphi(0) = 1$, that is $G_{1/3}(Pr) = -\mathrm{d}\varphi/\mathrm{d}\bar{\eta} \simeq 0.69124Pr^{1/20}$, for values of the Prandtl number close to unity [10]. We define $\theta_{w1}(\chi, z, \xi) = \chi^{1/3}\varphi_{w1}(z, \xi)$. The first order corrections for the nondimensional temperature in the wall are then

$$\left.\frac{\partial \varphi_{w1}}{\partial z}\right|_{z=-1} = 0 \quad \text{and} \quad \varphi_{w1}(0, \xi) = \left(\frac{H(\xi)}{G_{1/3}(Pr)}\right)^{5/6}. \tag{46}$$

Equations (46) are needed to solve the first order equation (42). The overall nondimensional thermal energy of the plate $\bar{\theta}_w = \int_0^1 \int_0^1 \theta_w \, d\chi \, dz$, for this limit up to the first order, takes the form

$$\bar{\theta}_w \approx \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\exp(-(n-1/2)^2 \pi^2 \xi)}{(n-1/2)^2} + \frac{3\alpha}{4\varepsilon^2} \int_0^1 \varphi_{w1} \, dz. \tag{47}$$

The leading term solution gives

$$\xi^* \simeq 0.4767 \quad \text{and} \quad \left.\frac{d\bar{\theta}_{w0}}{d\xi}\right|_{\xi^*} = -0.6165 \quad \text{for } \bar{\theta}_w = 0.25 \tag{48}$$

and is independent of the flow conditions (purely solid diffusion). The first order correction gives

$$\frac{\tau}{\tau^*} \simeq \frac{\xi^*}{\tau^*}\frac{\varepsilon^2}{\alpha} - \frac{3}{4\tau^*} \frac{I(Pr, \xi^*)}{\left.\frac{d\bar{\theta}_{wo}}{d\xi}\right|_{\xi^*}} \doteq 0.19595 \frac{\varepsilon^2}{\alpha} + 0.4053 \tag{49}$$

where $I(Pr, \xi^*) = \int_0^1 \varphi_{w1} \, dz \doteq 0.8105$ for $Pr = 1$.

5. RESULTS AND DISCUSSION

The system of equations (7) to (14) were integrated numerically using Keller's method [13] for the boundary layer equations, together with the technique of alternate directions from Peaceman, Rachford and Douglas for the heat conduction equation for the plate [14]. The boundary conditions in the fluid for $\eta \to \infty$ uses a finite mesh point, η_∞, chosen by making numerical experiments by increasing η_∞ until a non-significative change in the solution is obtained (for $Pr = 1$, $\eta_\infty = 9$ produces an error in the solution less than 1×10^{-10}). Because of the nonlinearity of the boundary layer equations, it was necessary to implement an iterative method based on the Newton's technique [13], with a convergence lower than 1×10^{-10}. The mesh used for the fluid balance equations were 40×90, in the longitudinal and transversal directions, respectively with a time step not larger than 0.001. For the solid we used a 90×50 grid. In all the calculations presented in this work we assumed for simplicitly a value of $Pr = 1$ and a uniform initial condition $\theta_w = 1$.

The most important parameter to be obtained in this work is the nondimensional evolution time. Figure 3 shows the nondimensional transient time τ needed to reach the value of $\bar{\theta}_w = 0.25$, as a function of α and different values of ε. The time is normalized with τ^*, that is the nondimensional time needed to reach the same global thermal energy for the case of $\alpha \to \infty$, $\tau^* \simeq 2.4328\ldots$. The asymptotic solution for the thermally thin wall regime is also plotted, showing how the transient evolution time is reduced always as α decreases in this regime. As the value of ε decreases, the minimum value of the normalized evolution time also decreases. This minimum value is produced in the thermally thin wall regime with negligible longitudinal heat conduction effects, $\varepsilon^2 \ll \alpha \ll 1$. The asymptotic solution for $\alpha \to \infty$, gives accurate results for values of α of order unity, as the value of ε decreases.

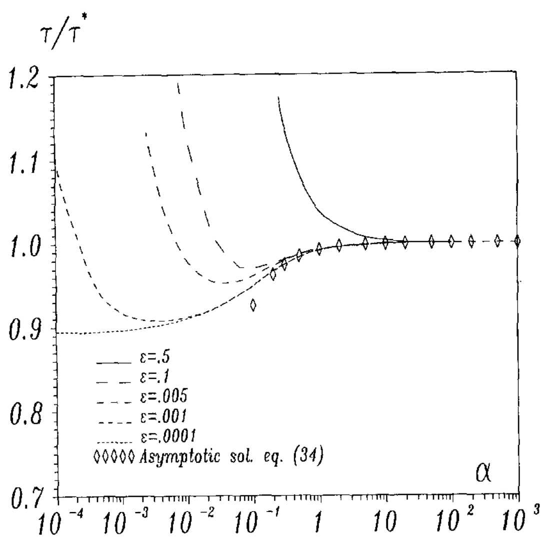

Fig. 3. Normalized nondimensional evolution time as a function of the nondimensional longitudinal thermal conductance of the plate α, for different values of the aspect ratio of the plate ε. The asymptotic solution for $\alpha \to \infty$ is also plotted with symbols $\diamond$.

Figure 4 shows the nondimensional evolution time as a function of α/ε^2. The two term asymptotic solution for the thermally thick wall regime (49) is also plotted. All curves for different values of ε show a universal character for $\alpha/\varepsilon^2 \to 0$ and $\alpha/\varepsilon^2 \to \infty$. However, in the middle region (transition from the thermally thin to thermally thick wall regimes) the solution depends strongly on the aspect ratio of the plate, ε. In all cases, we obtain a minimum value of the nondimensional evolution time in this middle region. However, this minimum value is also dependent strongly on ε.

For slightly different values of the Prandtl number (for example air), we can obtain the evolution cooling time by using the Taylor series expansion result

$$\tau(Pr) - \tau(Pr = 1) \approx 0.4509(Pr^{-1/20} - 1) + \frac{0.007981}{\alpha}(Pr - 1).$$

By way of illustration, we did some calculations in

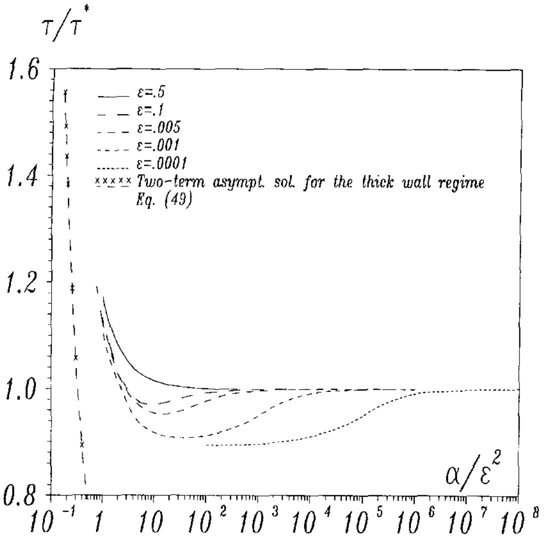

Fig. 4. Normalized nondimensional evolution time as a function of the nondimensional parameters α/ε^2, for different values of the aspect ratio of the plate ε. The two term asymptotic solution for the thermally thick wall regime $\alpha/\varepsilon^2 \to 0$ is also plotted with symbols $\times$.

physical units of the cooling time for a plate with different materials in air. We choose a temperature difference of 73 K, and the following plate dimensions: $h = 0.75$ cm and $L = 7.5$ cm. The Rayleigh number is then 2.893×10^6, which is large enough to produce a boundary layer but small enough to maintain the laminar flow. Figure 5 shows the resulting values of α, the nondimensional cooling time and the cooling time in physical units for an aluminum, silver, steel, lead and glass plates. For all the materials used for the plate, except glass, we obtain fairly large values of α and thus α/ε^2. The appropriate regime is then the thermally thin regime with large values of α. For the glass plate we are also in the thermally thin wall regime, but with values of α of order unity.

In summary, the most important conclusion from this work is related to the influence of the longitudinal heat conduction through the solid for the transient process. The nondimensional evolution time shows a minimum for finite values of the parameter α. This minimum appears in the thermally thin wall regime, decreasing its value as ε decreases. The transient process is then faster for the thermally thin wall regime without the effect of the longitudinal heat conduction ($\varepsilon^2 \ll \alpha \ll 1$).

Acknowledgement—This work has been supported by the research grant IN107795, DGAPA at UNAM, Mexico.

REFERENCES

1. Stewartson, K., On the free convection from a horizontal plate. *Z.A.M.P.*, 1958, **9a**, 276–282.
2. Gill, W. N., Zeh, D. W. and del Casal, E., Free convection on a horizontal plate. *Z.A.M.P.*, 1965, **16,** 539–541.
3. Rotem, Z., Free convection boundary layer flow over horizontal disks and plates I: similar solutions near plate. *Proc. 1st Canadian National Congress on Appl. Mech.*, 1967, **2b**, 309–310.
4. Rotem, Z. and Claassen, L., Natural convection above unconfined horizontal surfaces. *Journal of Fluid Mechanics*, 1969, **38**, 173–192.
5. Chellappa, A. K. and Singh, P., Possible similarity formulations for laminar free convection on a semi-infinite horizontal plate. *International Journal of Engineering Science*, 1989, **27**(2), 161–167.
6. Aihara, T., Yamada, Y. and Endo, S., Free convection along the downward-facing surface of a heated hori-

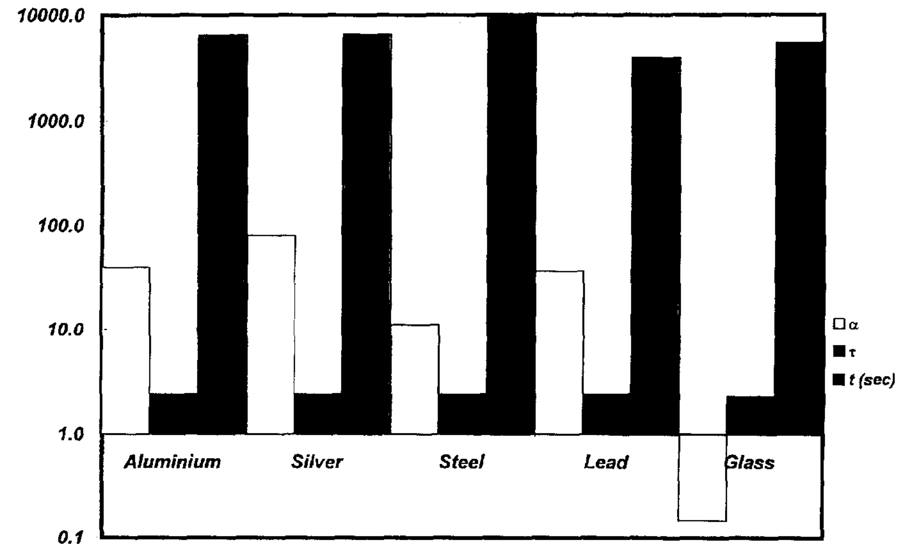

Fig. 5. Values of α, nondimensional and dimensional cooling time of aluminum, silver, steel lead and glass plates in air. The plate dimensions are $L = 7.5$ cm and $h = 0.75$ cm. The assumed temperature difference is 73 K.

zontal plate. *International Journal of Heat and Mass Transfer*, 1972, **15**, 2535–2549.

7. Schulenberg, T., Natural convection heat transfer below downward facing horizontal surfaces. *International Journal of Heat and Mass Transfer*, 1985, **28**, 467–477.
8. Higuera, F. J., Natural convection below a downward facing horizontal plate. *European Journal of Mechanics: B/Fluids*, 1993, **12**(3), 289–312.
9. Gebhart, B., Jaluria, Y., Mahajan, R. L. and Sammakia, B., *Buoyancy-Induced Flow and Transport*. Hemisphere, NY, 1988.
10. Pera, L. and Gebhart, B., Natural convection boundary layer flow over horizontal and slightly inclined surfaces. *International Journal of Heat and Mass Transfer*, 1973, **16**, 1131.
11. Luna, E., Treviño, C. and Higuera, F. J., Conjugate natural convection heat transfer between two fluids separated by a horizontal wall: steady-state analysis. *Heat and Mass Transfer*, 1996, **31**, 353–358.
12. Luna, E. and Treviño, C., Transient analysis of the natural laminar convection heat transfer process between two fluids separated by a thin horizontal wall. *European Journal of Mechanics: B/Fluids*, 1997, **16**, 429–446.
13. Cebeci, T. and Bradshaw, P., *Physical and Computational Aspects of Convective Heat Transfer*. Springer, Berlin, 1984.
14. Sod, A. Gary, *Numerical Methods in Fluid Dynamics: Initial and Initial Boundary-Value Problems*. Cambridge University Press, U.S.A., 1985.